CHARACTERISTIC CONES OF THE EQUATIONS IN THE NONLINEAR THEORY OF ELASTICITY

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UDC 539.3+517.945

The roots of the equation for the characteristic normals for two systems of differential equations in the nonlinear theory of elasticity are investigated. The first model is constructed using a thermodynamic identity. The second is a very simple hypoelastic model (the deviator of the stress-rate tensor is proportional to the deviator of the strain-rate tensor). It is shown that the roots of the equations for the normals to the characteristics for the second model are the same as the first-order terms in the expansion of the roots of the first model with respect to the strain-tensor deviator.

In this paper we study the characteristic cones of the differential equations describing two models in the nonlinear theory of elasticity. The first model, the differential equations of which are obtained on the basis of a certain thermodynamic identity, was formulated by Godunov and Romenskii [1]. The other model is widely used in calculating plastic–elastic flow [2]. The thermodynamics of this model does not have a satisfactory explanation. It can be treated as a certain approximation of the thermodynamics for the model [1].

The system of differential equations formulated in [1] has the form

$$\begin{gathered}\rho\frac{du_i}{dt}-\frac{\partial\sigma_{ik}}{\partial x_k}=0\\ \frac{dg_{ik}}{dt}+g_{i\alpha}\frac{\partial u_\alpha}{\partial x_k}+g_{k\alpha}\frac{\partial u_\alpha}{\partial x_i}=\varphi_{ik}(g_{mn},S),\qquad \frac{dS}{dt}=\varkappa(g_{mn},S)\end{gathered}\tag{1}$$

where u_i are the components of the velocity vector, σ_{ik} is the stress tensor, g_{ik} is the Cauchy deformation tensor, S is the entropy, and $\rho=\rho_0\sqrt{\det\|g_{ik}\|}$ is the density of the medium.

To close the system we give the dependence of σ_{ik} on g_{ik} and S by the Murnaghan equation

$$\sigma_{ik}=-2\rho\frac{\partial E}{\partial g_{i\alpha}}g_{\alpha k}$$

where $E=E(g_{mn},S)$ is the internal energy of the medium. We shall consider an isotropic medium, assuming that E depends on the three independent invariants of the strain tensor and on the entropy. It is convenient to consider the following dependences:

$$\begin{gathered}E=E(k_1,k_2,k_3,S),\ k_i=1/\sqrt{g_i}\\ E=E(\rho,D,\Delta,S)\\ \rho=\rho_0/(k_1k_2k_3),\ D={}^1/_2\,(d_1^2+d_2^2+d_3^2),\\ \Delta={}^1/_3\,(d_1^3+d_2^3+d_3^3)\\ d_i=\ln(k_i/\sqrt[3]{k_1k_2k_3})\quad(d_1+d_2+d_3=0)\end{gathered}$$

where k_i are the compression coefficients along the principal axes of the strain tensor.

The invariants k_1, k_2, and k_3 can be rewritten in terms of ρ, D, and Δ.

The equation, obtained by Godunov and Romenskii [1], for the characteristic normals has the form

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 3, pp. 126–132, May–June, 1974. Original article submitted October 24, 1973.

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$$\det(\Omega^2 I - \Lambda) = 0$$

$$\Lambda = \begin{pmatrix} L_1\eta_1^2 + M_3\eta_2^2 + M_2\eta_3^2 & N_3\eta_1\eta_2 & N_2\eta_1\eta_3 \\ N_3\eta_2\eta_1 & M_3\eta_1^2 + L_2\eta_2^2 + M_1\eta_3^2 & N_1\eta_2\eta_3 \\ N_2\eta_3\eta_1 & N_1\eta_3\eta_2 & M_2\eta_1^2 + M_1\eta_2^2 + L_3\eta_3^2 \end{pmatrix}$$

where $\eta_i = k_i\xi_i$, $\Omega = \omega + \dot{u}_\alpha\xi_\alpha$, and $(\omega, \xi_1, \xi_2, \xi_3)$ is the normal vector to the characteristic surface.

The nine elastic moduli – L_i, M_i, N_i – are expressed in terms of the first and second derivatives of the equation of state $E(k_1, k_2, k_3, S)$ according to the equations

$$L_1 = E_{k_1k_1}, \quad M_1 = \frac{k_2E_{k_2} - k_3E_{k_3}}{k_2^2 - k_3^2}, \quad N_1 = E_{k_2k_3} - \frac{k_2E_{k_3} - k_3E_{k_2}}{k_2^2 - k_3^2}$$

and the rest of the L_i, M_i, and N_i are obtained from these by the appropriate substitution of indices.

The matrix Λ corresponds to a crystal of the rhombic system [3], whose anisotropy is described by the nine independent elastic moduli. In such a medium sound waves propagate in three directions, generally speaking, with different velocities. Fedorov [3] discusses the theory of elastic waves in crystals; however, there only small deformations are studied and the calculation of the characteristics is not carried through to completion. In the present case knowing interpolation equations for the equation of state, we can elucidate the nature of sound-wave propagation. We shall determine how sound waves propagate with the accuracy of the first terms in the expansion of the elastic moduli with respect to the strain-tensor deviator in a neighborhood of zero deviator for any density and entropy.

Using the parameterization of the equation of state $E = E(\rho, D, \Delta, S)$, we can expand the matrix Λ in powers of the strain deviator d_i in a neighborhood of $d_i = 0$:

$$\Lambda = \Lambda^{(0)}(\rho, S) + d_i\Lambda_i^{(1)}(\rho, S) + d_i^2\Lambda_i^{(2)}(\rho, S) + \ldots$$

and study the eigenvalues of Λ as perturbations of the eigenvalues of $\Lambda^{(0)}$.

We restrict ourselves to the first-order perturbation. In order to calculate the elastic moduli we can use Eqs. (5.2) from [1] to write the matrices

$$\Lambda^{(0)} = \begin{pmatrix} (l+m)\xi_1^2 + m(\xi_1^2+\xi_2^2+\xi_3^2) & (l+m)\xi_1\xi_2 & (l+m)\xi_1\xi_3 \\ (l+m)\xi_2\xi_1 & (l+m)\xi_2^2 + m(\xi_1^2+\xi_2^2+\xi_3^2) & (l+m)\xi_2\xi_3 \\ (l+m)\xi_3\xi_1 & (l+m)\xi_3\xi_2 & (l+m)\xi_3^2 + m(\xi_1^2+\xi_2^2+\xi_3^2) \end{pmatrix}$$

$$\Lambda_1^{(1)} = \begin{pmatrix} -2(2m+2k-n)\xi_1^2 + (m+{}^3/_2 n)(\xi_2^2+\xi_3^2) & 0 & 0 \\ 0 & (m+{}^3/_2 n)\xi_1^2 & (m+{}^1/_2 n+2k)\xi_2\xi_3 \\ 0 & (m+{}^1/_2 n+2k)\xi_3\xi_2 & (m+{}^3/_2 n)\xi_1^2 \end{pmatrix}$$

$$\Lambda_2^{(1)} = \begin{pmatrix} (m+{}^3/_2 n)\xi_2^2 & 0 & (m+{}^1/_2 n+2k)\xi_1\xi_3 \\ 0 & -2(2m+2k-n)\xi_2^2 + (m+{}^3/_2 n)(\xi_1^2+\xi_3^2) & 0 \\ (m+{}^1/_2 n+2k)\xi_3\xi_1 & 0 & (m+{}^3/_2 n)\xi_2^2 \end{pmatrix}$$

$$\Lambda_3^{(1)} = \begin{pmatrix} (m+{}^3/_2 n)\xi_3^2 & (m+{}^1/_2 n+2k)\xi_1\xi_2 & 0 \\ (m+{}^1/_2 n+2k)\xi_2\xi_1 & (m+{}^3/_2 n)\xi_3^2 & 0 \\ 0 & 0 & -2(2m+2k-n)\xi_3^2 + (m+{}^3/_2 n)(\xi_1^2+\xi_2^2) \end{pmatrix},$$

$$l = (\rho^2E_\rho)_\rho - {}^1/_3 E_D, \quad m = {}^1/_2 E_D, \quad n = {}^1/_3 E_\Delta, \quad k = {}^1/_2 E_{\rho D}$$

where all quantities are calculated at the point $d_1 = d_2 = d_3 = 0$. The matrices $\Lambda_i^{(1)}$ admit some arbitrariness in the way they are written, since $d_1 + d_2 + d_3 = 0$; their specific form is chosen from considerations of symmetry.

It is necessary to explain how the eigenvalues of $\Lambda^{(0)}$ are perturbed by the perturbed matrix $d_i\Lambda_i^{(1)}$. The eigenvalues of a multiparameter perturbation might not be differentiable with respect to the parameters of the perturbation [4]. Therefore, we go to a one-parameter perturbation

$$\Lambda = \Lambda^{(0)} + \varepsilon(d_i\varepsilon^{-1}\Lambda_i^{(1)}) + O(D)$$

where $\varepsilon = \sqrt{D}$. We denote $\Lambda^{(1)} = \varepsilon^{-1}d_i\Lambda_i^{(1)}$.

The eigenvalues for a one-parameter symmetrical perturbation are real and differentiable with respect to the perturbation parameter [4]. This makes it possible to determine the terms in the expansion of the eigenvalues and eigenvectors in powers of ε. Let us consider the first terms in the expansion of the eigenvalues.

The matrix $\Lambda^{(0)}$ has one simple and one twofold eigenvalue

$$(\Omega_1^2)^{(0)} = (l + 2m)(\xi_1^2 + \xi_2^2 + \xi_3^2), \; (\Omega_2^2)^{(0)} = (\Omega_3^2)^{(0)} = m(\xi_1^2 + \xi_2^2 + \xi_3^2)$$

To these eigenvalues corresponds an orthonormal system of eigenvectors, e.g.,

$$e_1 = \frac{1}{\sqrt{\xi_1^2+\xi_2^2+\xi_3^2}}\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\end{pmatrix}, \quad e_2 = \frac{1}{\sqrt{\xi_1^2+\xi_2^2}}\begin{pmatrix}\xi_2\\ -\xi_1\\ 0\end{pmatrix},$$
$$e_3 = \frac{1}{\sqrt{(\xi_1^2+\xi_2^2)(\xi_1^2+\xi_2^2+\xi_3^2)}}\begin{pmatrix}-\xi_1\xi_3\\ -\xi_2\xi_3\\ \xi_1^2+\xi_2^2\end{pmatrix}$$

Denoting by Ω_1^2, Ω_2^2, Ω_3^2 the eigenvalues of Λ and expanding them in powers of ε, we can show that

$$(\Omega_1^2)^{(1)} = (\Lambda^{(1)} e_1, e_1)$$

and $(\Omega_2^2)^{(1)}$ and $(\Omega_3^2)^{(1)}$ are eigenvalues of the matrix

$$\begin{pmatrix}(\Lambda^{(1)}e_2, e_2) & (\Lambda^{(1)}e_2, e_3)\\ (\Lambda^{(1)}e_3, e_2) & (\Lambda^{(1)}e_3, e_3)\end{pmatrix}$$

The matrix $\Lambda^{(1)}$ and the vectors e_1, e_2, e_3 are known, hence, we find

$$(\Lambda^{(1)} e_1, e_1) = -2(m + 2k - {}^1/_2 n)(d_1\xi_1^2 + d_2\xi_2^2 + d_3\xi_3^2)/\sqrt{D}$$
$$(\Lambda^{(1)} e_2, e_2) = ({}^3/_2 n + m)(d_1\xi_1^2 + d_2\xi_2^2 + d_3\xi_3^2)/\sqrt{D} -$$
$$-({}^3/_2 n - m)(\xi_1^2 + \xi_2^2 + \xi_3^2)d_3/\sqrt{D} - ({}^3/_2 n - m)(\xi_1^2 + \xi_2^2 + \xi_3^2)(d_1\xi_1^2 + d_2\xi_2^2)/(\xi_1^2 + \xi_2^2)\sqrt{D}$$
$$(\Lambda^{(1)} e_2, e_3) = (\Lambda^{(1)} e_3, e_2) = ({}^3/_2 n - m)\xi_1\xi_2\xi_3(d_2 - d_1)\sqrt{\xi_1^2 + \xi_2^2 + \xi_3^2}/(\xi_1^2 + \xi_2^2)\sqrt{D}$$
$$(\Lambda^{(1)} e_3, e_3) = ({}^3/_2 n + m)(d_1\xi_1^2 + d_2\xi_2^2 + d_3\xi_3^2)/\sqrt{D} +$$
$$+({}^3/_2 n - m)(\xi_1^2 + \xi_2^2)d_3/\sqrt{D} + ({}^3/_2 n - m)\xi_3^2(d_1\xi_1^2 + d_2\xi_2^2)/(\xi_1^2 + \xi_2^2)\sqrt{D}$$

Calculating $(\Omega_i^2)^{(1)}$ by the indicated method, we obtain

$$\Omega_1^2 = (l + 2m)(\xi_1^2 + \xi_2^2 + \xi_3^2) - 2(m + 2k - {}^1/_2 n)(d_1\xi_1^2 + d_2\xi_2^2 + d_3\xi_3^2) + O(D)$$
$$\Omega_2^2 = m(\xi_1^2 + \xi_2^2 + \xi_3^2) + {}^3/_2(m + {}^1/_2 n)(d_1\xi_1^2 + d_2\xi_2^2 + d_3\xi_3^2) - {}^1/_2(m - {}^3/_2 n)\,[(d_1\xi_1^2 + d_2\xi_2^2 + d_3\xi_3^2)^2 - 4(\xi_1^2 + \xi_2^2 +$$
$$+ \xi_3^2)(d_2d_3\xi_1^2 + d_1d_3\xi_2^2 + d_1d_2\xi_3^2)]^{1/2} + O(D) \quad \Omega_3^2 = m(\xi_1^2 + \xi_2^2 + \xi_3^2) + {}^3/_2(m + {}^1/_2 n)(d_1\xi_1^2 + d_2\xi_2^2 +$$
$$+ d_3\xi_3^2) + {}^1/_2(m - {}^3/_2 n)[(d_1\xi_1^2 + d_2\xi_2^2 + d_3\xi_3^2)^2 - 4(\xi_1^2 + \xi_2^2 + \xi_3^2)(d_2d_3\xi_1^2 + d_1d_3\xi_2^2 + d_1d_2\xi_3^2)]^{1/2} + O(D)$$

Hence, it is seen that the perturbed eigenvalues for $(\Omega_2^2)^{(0)} = (\Omega_3^2)^{(0)}$ differ by a magnitude

$$(m - {}^3/_2 n)[(d_1\xi_1^2 + d_2\xi_2^2 + d_3\xi_3^2) - 4(\xi_1^2 + \xi_2^2 + \xi_3^2)(d_2d_3\xi_1^2 + d_1d_3\xi_2^2 + d_1d_2\xi_3^2)]^{1/2} + O(D)$$

The expression under the square root sign is always nonnegative. For example, if $d_1 \le d_2$ and $d_1 \le d_3$, then it can be rewritten in the form

$$[(d_2 - d_3)\xi_1^2 + (d_1 - d_3)\xi_2^2 + (d_2 - d_1)\xi_3^2]^2 + 4(d_2 - d_1)(d_3 - d_1)\xi_2^2\xi_3^2 \geqslant 0$$

The equality with zero is achieved when $d_1 = d_2 = d_3$.

Let us consider the interpolation equations for the equation of state $E(\rho, D, \Delta, S)$. From our calculations it follows that at the point $d_1 = d_2 = d_3 = 0$

$$\frac{\partial}{\partial\rho}(\rho^2 E_\rho) = l + {}^2/_3 m = (l + 2m) - {}^4/_3 m = c_0^2 - {}^4/_3 c_1^2$$
$${}^1/_2\,\partial E/\partial D = m = c_1^2$$

where c_0 and c_1 are the propagation velocities of longitudinal and transverse sound waves. In order to describe the simplest nonlinear effects of sound-wave propagation, it suffices to give the equation of state in the form

$$E(\rho, D, S) = E^{(0)}(\rho, S) + 2m(\rho, S)D$$

We consider another model. The system of differential equations describing this model includes hypoelastic relations as the stress–strain coupling. These relations with the addition of dissipative terms are a particular case of the Reuss equations of plasticity theory [5]. The closed system of equations has the form

$$\frac{d\rho}{dt} = -\rho\frac{\partial u_i}{\partial x_j}\delta_{ij}, \quad \rho\frac{dE}{dt} = -p\frac{\partial u_i}{\partial x_j}\delta_{ij} + \sigma_{ik}'\frac{\partial u_i}{\partial x_k} \tag{2}$$

$$\rho\frac{du_i}{dt} + \frac{\partial p}{\partial x_i} - \frac{\partial \sigma_{ik}'}{\partial x_k} = 0$$

$$\frac{d\sigma_{ik}'}{dt} = -\frac{1}{2}\sigma_{i\alpha}'\left(\frac{\partial u_\alpha}{\partial x_k} - \frac{\partial u_k}{\partial x_\alpha}\right) - \frac{1}{2}\sigma_{k\alpha}'\left(\frac{\partial u_\alpha}{\partial x_i} - \frac{\partial u_i}{\partial x_\alpha}\right) + \mu\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\frac{\partial u_\alpha}{\partial x_\beta}\delta_{\alpha\beta}\delta_{ik}\right)$$

Here ρ is the density, E is the internal energy, p is the pressure, u_i are the components of the velocity, σ_{ik}' is the deviator of the stress tensor, and μ is a function of ρ, E, and σ_{ik}'.

To close the system a relation $p = p(\rho, E, \sigma_{ik}')$ is given. Let us consider the relation $p = p(\rho, E)$, which was used by Wilkins [2] in numerical calculations. To calculate the characteristic matrix we use the method of reducing the system of differential equations to a system of second-order equations which was used in [1]. This procedure consists of extending the initial system by differentiating with respect to t and x_i. In doing this the vertical characteristics (lines of flow) are isolated, and the part of the characteristic matrix associated with the main terms in the velocity equations correspond to the three propagation velocities of sound waves.

We apply the operator d/dt to the equations for u_i:

$$\rho\frac{d^2u_i}{dt^2} + p_E\frac{\partial}{\partial x_i}\left(\frac{dE}{dt}\right) + p_\rho\frac{\partial}{\partial x_i}\left(\frac{d\rho}{dt}\right) - \frac{\partial}{\partial x_k}\left(\frac{d\sigma_{ik}'}{dt}\right) + \ldots = 0$$

The dots here and henceforth denote terms of the equations which are inessential for calculating the characteristics. They contain no derivatives higher than first order. From the remaining equations we find

$$\frac{\partial}{\partial x_i}\left(\frac{dE}{dt}\right) = -\frac{p}{\rho}\frac{\partial^2 u_k}{\partial x_i\partial x_k} + \frac{\sigma_{jk}'}{\rho}\frac{\partial^2 u_j}{\partial x_i\partial x_k} + \ldots, \quad \frac{\partial}{\partial x_i}\left(\frac{d\rho}{dt}\right) = -\rho\frac{\partial^2 u_k}{\partial x_i\partial x_k} + \ldots$$

$$\frac{\partial}{\partial x_k}\left(\frac{d\sigma_{ik}'}{dt}\right) = -\frac{1}{2}\sigma_{i\alpha}'\left(\frac{\partial^2 u_\alpha}{\partial x_k\partial x_k} - \frac{\partial^2 u_k}{\partial x_\alpha\partial x_k}\right) - \frac{1}{2}\sigma_{k\alpha}'\left(\frac{\partial^2 u_\alpha}{\partial x_i\partial x_k} - \frac{\partial^2 u_i}{\partial x_\alpha\partial x_k}\right) + \mu\left(\frac{\partial^2 u_i}{\partial x_k\partial x_k} + \frac{\partial^2 u_k}{\partial x_i\partial x_k} - \frac{2}{3}\delta_{ik}\delta_{\alpha\beta}\frac{\partial^2 u_\alpha}{\partial x_\beta\partial x_k}\right) + \ldots$$

Putting the obtained expressions into the equations for d^2u_i/dt^2 and after grouping together the terms, we find

$$\rho\frac{d^2u_i}{dt^2} - \left(\rho p_\rho + \frac{1}{\rho}p_E p + \frac{1}{3}\mu\right)\frac{\partial^2 u_k}{\partial x_i\partial x_k} + \left(\frac{1}{2} + \frac{1}{\rho}p_E\right)\sigma_{kj}'\frac{\partial^2 u_j}{\partial x_i\partial x_k} - \mu\frac{\partial^2 u_i}{\partial x_k\partial x_k} + \frac{1}{2}\sigma_{ij}'\frac{\partial^2 u_j}{\partial x_j\partial x_k} - \frac{1}{2}\sigma_{kj}'\frac{\partial^2 u_i}{\partial x_j\partial x_k} + \ldots = 0$$

If we denote the elements of the characteristic matrix by Λ_{ij}, then the equation for the characteristic normals takes the form

$$\det(\rho\Omega^2 I - \Lambda) = 0, \quad \Lambda = \|\Lambda_{ij}\|$$

where Λ_{ij} are found according to the equation

$$\Lambda_{ij} = \left(\rho p_\rho + \frac{1}{\rho}p_E p + \frac{1}{3}\mu\right)\xi_i\xi_j - \left(\frac{1}{2} + \frac{1}{\rho}p_E\right)\sigma_{kj}'\xi_i\xi_k + \frac{1}{2}\sigma_{ik}'\xi_k\xi_j - \frac{1}{2}\sigma_{ij}'\xi_k\xi_k + \mu\xi_k\xi_k\delta_{ij} + \frac{1}{2}\sigma_{kl}'\xi_k\xi_l\delta_{ij}$$

Here $\Omega = \omega + u_i\xi_i$ and $(\omega, \xi_1, \xi_2, \xi_3)$ is the normal vector to the characteristic surface.

The matrix Λ is asymmetric, and hence we cannot immediately conclude that its eigenvalues are real. However, they can be written out in terms of explicit equations. Consider the sum $\xi_j\Lambda_{ij}$ (in matrix terms this means that to the first column of Λ, multiplied by ξ_1, are added the second and third columns, multiplied, respectively, by ξ_2 and ξ_3)

$$\xi_j\Lambda_{ij} = \xi_i[(\rho p_\rho + \rho^{-1}p_E p + {}^4/_3\mu)\xi_k\xi_k - \rho^{-1}p_E\sigma_{kl}'\xi_k\xi_l]$$

The elements of the first column in the characteristic determinant after such a transformation turn out to be proportional, and hence the characteristic equation splits into two equations:

$$\rho\Omega^2 = (\rho p_\rho + \rho^{-1}p_E p + {}^4/_3\mu)\xi_k\xi_k - \rho^{-1}p_E\sigma'_{ij}\xi_i\xi_j$$

$$\begin{vmatrix} \xi_1 & -\Lambda_{12} & -\Lambda_{13} \\ \xi_2 & \rho\Omega^2-\Lambda_{22} & -\Lambda_{23} \\ \xi_3 & -\Lambda_{32} & \rho\Omega^2-\Lambda_{33} \end{vmatrix} = 0$$

The first equation is an explicit equation for one root, corresponding to the propagation velocity of longitudinal waves. The second equation, quadratic with respect to $\rho\Omega^2$, has roots corresponding to the two propagation velocities of transverse waves.

Let us calculate the roots of the quadratic equation. We go over to the principal axes of the stress tensor $\sigma_{ii}'=\sigma_i'$, $\sigma_{ij}'=0$ $(i\neq j)$. The following Λ_{ij} are necessary:

$$\Lambda_{12}=\left(\rho p_\rho+\frac{1}{\rho}p_E p+\frac{1}{3}\mu\right)\xi_1\xi_2-\left(\frac{1}{2}+\frac{1}{\rho}p_E\right)\sigma_{22}'\xi_1\xi_2+\frac{1}{2}\sigma_{11}'\xi_1\xi_2$$

$$\Lambda_{13}=\left(\rho p_\rho+\frac{1}{\rho}p_E p+\frac{1}{3}\mu\right)\xi_1\xi_3-\left(\frac{1}{2}+\frac{1}{\rho}p_E\right)\sigma_{33}'\xi_1\xi_3+\frac{1}{2}\sigma_{11}'\xi_1\xi_3$$

$$\Lambda_{22}=\left(\rho p_\rho+\frac{1}{\rho}p_E p+\frac{1}{3}\mu\right)\xi_2^2+\mu\left(\xi_1^2+\xi_2^2+\xi_3^2\right)-\frac{1}{\rho}p_E\sigma_{22}'\xi_2^2+\frac{1}{2}\sigma_{11}'\xi_1^2+\frac{1}{2}\sigma_{22}'\xi_2^2+\frac{1}{2}\sigma_{33}'\xi_3^2-\frac{1}{2}\sigma_{22}'\left(\xi_1^2+\xi_2^2+\xi_3^2\right)$$

$$\Lambda_{23}=\left(\rho p_\rho+\frac{1}{\rho}p_E p+\frac{1}{3}\mu\right)\xi_2\xi_3-\left(\frac{1}{2}+\frac{1}{\rho}p_E\right)\sigma_{33}'\xi_2\xi_3+\frac{1}{2}\sigma_{22}'\xi_2\xi_3$$

$$\Lambda_{32}=\left(\rho p_\rho+\frac{1}{\rho}p_E p+\frac{1}{3}\mu\right)\xi_3\xi_2-\left(\frac{1}{2}+\frac{1}{\rho}p_E\right)\sigma_{22}'\xi_3\xi_2+\frac{1}{2}\sigma_{33}'\xi_3\xi_2$$

$$\Lambda_{33}=\left(\rho p_\rho+\frac{1}{\rho}p_E p+\frac{1}{3}\mu\right)\xi_3^2+\mu\left(\xi_1^2+\xi_2^2+\xi_3^2\right)-\frac{1}{\rho}p_E\sigma_{33}'\xi_3^2+\frac{1}{2}\sigma_{11}'\xi_1^2+\frac{1}{2}\sigma_{22}'\xi_2^2+\frac{1}{2}\sigma_{33}'\xi_3^2-\frac{1}{2}\sigma_{33}'\left(\xi_1^2+\xi_2^2+\xi_3^2\right)$$

Expanding the determinant, we obtain a quadratic equation:

$$\begin{aligned}&[\rho\Omega^2-\mu(\xi_1^2+\xi_2^2+\xi_3^2)]^2-{}^3/_2(\sigma_1'\xi_1^2+\sigma_2'\xi_2^2+\sigma_3'\xi_3^2)[\rho\Omega^2-\\&-\mu(\xi_1^2+\xi_2^2+\xi_3^2)]+{}^1/_4(\sigma_2'-\sigma_1')(\sigma_3'-\sigma_1')\xi_1^4+{}^1/_4(\sigma_1'-\\&-\sigma_2')(\sigma_3'-\sigma_2')\xi_2^4+{}^1/_4(\sigma_2'-\sigma_3')(\sigma_1'-\sigma_3')\xi_3^4-{}^1/_4(\sigma_2'-\\&-\sigma_1')^2\xi_1^2\xi_2^2-{}^1/_4(\sigma_3'-\sigma_1')^2\xi_1^2\xi_3^2-{}^1/_4(\sigma_3'-\sigma_2')^2\xi_2^2\xi_3^2=0\end{aligned}$$

The roots of this equation are

$$\begin{aligned}\rho\Omega_{2,3}^2=\mu(\xi_1^2+\xi_2^2+\xi_3^2)+{}^3/_4(\sigma_1'\xi_1^2+\sigma_2'\xi_2^2+\sigma_3'\xi_3^2)\pm\\\pm{}^1/_4[(\sigma_1'\xi_1^2+\sigma_2'\xi_2^2+\sigma_3'\xi_3^2)-4(\xi_1^2+\xi_2^2+\xi_3^2)(\sigma_2'\sigma_3'\xi_1^2+\sigma_3'\sigma_1'\xi_2^2+\sigma_1'\sigma_2'\xi_3^2)]^{1/2}\end{aligned}$$

They correspond to the propagation velocities of transverse waves. The root, corresponding to the propagation velocity of longitudinal waves, in the principal axes has the form

$$\rho\Omega_1^2=(\rho p_\rho+\rho^{-1}p_E p+{}^4/_3\mu)(\xi_1^2+\xi_2^2+\xi_3^2)-\rho^{-1}p_E(\sigma_1'\xi_1^2+\sigma_2'\xi_2^2+\sigma_3'\xi_3^2)$$

If we set $\sigma_i'=2\mu d_i$, then the calculated roots with the specified accuracy coincide with the approximations to the eigenvalues calculated for model (1). This serves as a confirmation of the fact that the model (2) considered here is an approximation of model (1) with the equation of state $E=E^{(0)}(\rho, s)+2m(\rho, S)D$. The defect of model (2) lies in the lack of any integral, i.e., the conservation of entropy in adiabatic processes.

The author is grateful to S. K. Godunov for discussions.

LITERATURE CITED

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